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An extended q-deformed $su(2)$ algebra and the Bloch electron problem

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Abstract

It is shown that an extended q-deformed $su(2)$ algebra with an extra ("Schwinger ") term can describe Bloch electrons in a uniform magnetic field with an additional periodic potential. This is a generalization of the analysis of Bloch electrons by Wiegmann and Zabrodin. By using a representation theory of this q-deformed algebra, we obtain functional Bethe ansatz equations whose solutions should be functions of finite degree. It is also shown that the zero energy solution is expressed in terms of an Askey Wilson polynomial.

The quantum deformation [1] of algebras was introduced in connection with inverse scattering problems and integrable models. It is also known that this notion has a deep relation to Yang-Baxter equations [2]. The $U_q(sl_2)$ algebra, one of the q-deformed algebras, was applied to an analysis of the two-dimensional Bloch electron problem [3] by Wiegmann and Zabrodin [4]. They found functional Bethe ansatz equations for this problem. From the representation theory of $U_q(sl_2)$ algebra on the functional space, they deduce that the solutions of the Bethe ansatz equations should be functions of finite degree. The zero energy solutions and the generalized functional Bethe ansatz equations have also been discussed in Refs. [5][6]. Biedenharn [7] and Macfarlane [8] constructed $U_q(sl_2)$ algebra, in the manner of Schwinger's construction of conventional $su(2)$, by using two sets of q-deformed oscillator algebras. The q-deformed oscillator algebra $\mathcal{H}_q(1)$, if suitably defined,

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supports the Hopf structure [9] [10]. A method to construct a representation of $\mathcal{H}_q(1)$, which is manifestly free of negative norm, was proposed [11]. This representation of $\mathcal{H}_q(1)$ was used to analyze the phase operator problem [12] [13] of the photon by using a notion of index [14].

Recently, an extended q-deformed $su(2)$ algebra was proposed [15] by using a representation of the above q-deformed oscillator algebra, which is manifestly free of negative norm. This construction is a one parameter generalization (the Casimir degree of freedom "n_0") of the construction of Biedenharn [7] and Macfarlane [8]. It is reduced to conventional $U_q(sl_2)$ if one fixes the Casimir to a specific value. This algebra has an additional term in commutation relations of $U_q(sl_2)$, which we tentatively called as " Schwinger term", since this extra term was originally introduced to maintain the representation free of negative norm for the deformation parameter q at a root of unity [15]. This extended q-deformed $su(2)$ algebra can be realized on a functional space which is analogous to the conventional functional realization of $U_q(sl_2)$, but it contains an additional one parameter degree of freedom corresponding to the Casimir freedom . In a manner similar to the case of $U_q(sl_2)$ algebra, we can construct a cyclic representation from the functional realization of the extended q-deformed $su(2)$ algebra. The non-linear correspondence[16] between this algebra and q-deformed oscillator algebra was found in their cyclic representations and functional realizations [17] [18]. Other aspects of q-deformed oscillator algebra and the extended q-deformed $su(2)$ algebra have also been discussed in Ref [19].

In this paper, we show that the extended q-deformed $su(2)$ algebra is used to describe a one-parameter generalization of the Wiegmann and Zabrodin analysis of the Bloch electron problem, in which an extra periodic potential appears in addition to the uniform magnetic field. Firstly, we summarize the notations of the extended q-deformed $su(2)$ algebra with a "Schwinger term". Defining relations of the algebra are,

$$\begin{aligned} [S_\pm, S_3] &= \mp S_\pm, \\ [S_+, S_-] &= [2S_3] + 4[n_0]\sin\pi\theta\sin 2\pi\theta[S_3][\mathcal{C} + \frac{1}{2}], \end{aligned} \tag{1}$$

where

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}. \tag{2}$$

The operator $\mathcal{C}$ commutes with all other quantities, and $\mathcal{C}$ is fixed at $2\mathcal{C}+1 = 2j+1-2n_0$

for a $2j+1$ dimensional representation of this algebra [15]. We note that the algebra (1) for

$$q = e^{2\pi i\theta}, \tag{3}$$

with an arbitrary real θ and for an arbitrary number n_0 has a $2j+1$ dimensional highest weight representation.

The $2j+1$ dimensional highest weight representation of the algebra (1) can also be realized by $q-$difference equations as

$$\begin{aligned}
\tilde{S}_+\psi(z) &= (q-q^{-1})^{-1}z(q^{2j-n_0}\psi(q^{-1}z) - q^{-2j+n_0}\psi(qz)) + z[n_0]\psi(z), \\
\tilde{S}_-\psi(z) &= -(q-q^{-1})^{-1}z^{-1}(q^{n_0}\psi(q^{-1}z) - q^{-n_0}\psi(qz)) + z^{-1}[n_0]\psi(z), \\
q^{\tilde{S}_3}\psi(z) &= q^{-j}\psi(qz),
\end{aligned} \tag{4}$$

where $\psi(z)$ is a polynomial of degree $2j$. This representation satisfies the highest weight condition $\tilde{S}_+ z^{2j} = 0$ and the lowest weight condition $\tilde{S}_- \cdot 1 = 0$. The representation of (4) for the bases, $z^{j+m},\ m = j, j-1, ..., -j$, is given by

$$\begin{aligned}
\tilde{S}_+ z^{j+m} &= ([j-m-n_0] + [n_0])z^{j+m+1}, \\
\tilde{S}_- z^{j+m} &= ([j+m-n_0] + [n_0])z^{j+m-1}, \\
q^{\tilde{S}_3} z^{j+m} &= q^m z^{j+m}.
\end{aligned} \tag{5}$$

This representation is related to the standard $2j+1$ dimensional representation $S_\pm, S_3$ by a non-unitary similarity transformation $S_\pm = A^{-1}\tilde{S}_\pm A, S_3 = \tilde{S}_3$ [15]. We will come back to this point later. For a specific deformation parameter $q = \exp[\pi i P/Q]$ for mutually co-prime integers P and Q , we can define a cyclic representation, which is obtained by putting $z = q^k, (k = 1, 2, \cdots, 2Q)$, in (4). There are $2Q$ bases $\psi_k \equiv \psi(q^k)$ which satisfy $\psi_{k+2Q} = \psi_k$, and we have

$$\begin{aligned}
\rho(S_+)\psi_k &= (q-q^{-1})^{-1}(-q^{k-2j+n_0}\psi_{k+1} + q^{k+2j-n_0}\psi_{k-1}) + q^k[n_0]\psi_k, \\
\rho(S_-)\psi_k &= (q-q^{-1})^{-1}(q^{-k-n_0}\psi_{k+1} - q^{-k+n_0}\psi_{k-1}) + q^{-k}[n_0]\psi_k, \\
q^{\rho(S_3)}\psi_k &= q^{-j}\psi_{k+1}.
\end{aligned} \tag{6}$$

It is confirmed that this cyclic representation $\rho(S)$ satisfies the algebra (1). Note that $2j+1$, P and Q are independent integers. In the original construction in [15], we chose

n_0 such that

$$[n_0] = \frac{\sin \pi n_0 \frac{P}{Q}}{\sin \pi \frac{P}{Q}} = \frac{1}{|\sin \pi \frac{P}{Q}|}, \tag{7}$$

to avoid the negative norm in the standard $2j+1$ dimensional representation.

We now consider tight-binding two-dimensional electrons in a magnetic field with an additional periodic potential

$$H \quad = \quad V_1 T_x + V_1 T_x^\dagger + V_2 T_y + V_2 T_y^\dagger + V_3 U + V_3^* U^\dagger, \tag{8}$$

where the parameters V_1 and V_2 are real, and V_3 is a complex number.

$$T_x = \sum_{m,n} c_{m+1,n}^\dagger c_{m,n} e^{i\theta_{m,n}^x}, \quad T_y = \sum_{m,n} c_{m,n+1}^\dagger c_{m,n} e^{i\theta_{m,n}^y}, \quad U = \sum_{m,n} q^{m+n} c_{m,n}^\dagger c_{m,n}. \tag{9}$$

Here $c_{m,n}$ is the annihilation operator for an electron at site (m,n), and we choose a very specific diagonal potential U containing the parameter q for later convenience. We assumed that the constant phase factors of V_1 and V_2 can be absorbed into the gauge potentials $\theta_{m,n}^x$ and $\theta_{m,n}^y$. The gauge potentials $\theta_{m,n}^x$ and $\theta_{m,n}^y$ are related to a flux per plaquette $\phi_{m,n}$ at (m,n) by

$$rot_{m,n}\theta = \Delta_x \theta_{m,n}^y - \Delta_y \theta_{m,n}^x = 2\pi\phi_{m,n}, \tag{10}$$

where the difference operators Δ_x and Δ_y operate on a lattice function $f_{m,n}$ as

$$\Delta_x f_{m,n} = f_{m+1,n} - f_{m,n}, \quad \Delta_y f_{m,n} = f_{m,n+1} - f_{m,n}. \tag{11}$$

The Hamiltonian (8) is invariant under the following gauge transformation,

$$c_i \to \Omega_i c_i, \quad e^{i\theta_{i,j}} \to \Omega_i e^{i\theta_{i,j}} \Omega_j^{-1}, \quad |\Omega_i| = 1. \tag{12}$$

In the following, we assume that the magnetic field is uniform and rational, $-\phi_{m,n} = \phi = P/Q$ with mutually prime integers P and Q. We can then take a diagonal gauge specified by

$$\theta_{m,n}^x = \pi\phi(m+n), \quad \theta_{m,n}^y = -\pi\phi(m+n+1). \tag{13}$$

In this diagonal gauge, if we choose the parameter q, which is later identified with the deformation parameter of q-deformed algebra, as a root of unity

$$q^{2Q} = 1, \quad q = e^{i\pi\frac{P}{Q}}, \quad P, Q \in Z, \tag{14}$$

the Hamiltonian becomes periodic along $(1,1)$ and $(1,-1)$ directions; the period in $(1,1)$ direction is $2Q$. We can then use the Bloch theorem with momenta p_+ and p_- for *one particle* states

$$|\Phi(p_+,p_-)\rangle = \sum_{m,n} \Psi_{m,n}(p_+,p_-)c^\dagger_{m,n}|0\rangle, \tag{15}$$

where

$$\Psi_{m,n}(\mathbf{p}) = e^{ip_-(m-n)+ip_+(m+n)}\psi_{m+n}(\mathbf{p}), \tag{16}$$

$$\psi_{k+2Q}(\mathbf{p}) = \psi_k(\mathbf{p}), \quad k = 1,\cdots,2Q. \tag{17}$$

Note that the "periodic potentials" in the present problem (8) depend only on the combination $m+n$ which is the reason why we have $\psi_{m+n}(\mathbf{p})$. The Schrödinger equation, $H|\Phi(\mathbf{p})\rangle = E|\Phi(\mathbf{p})\rangle$, is written on the basis ψ_k as follows

$$(V_1 q^{k-1}e^{-i(p_-+p_+)} + V_2 q^{-k}e^{i(p_--p_+)})\psi_{k-1} + (V_1 q^{-k}e^{i(p_-+p_+)} + V_2 q^{k+1}e^{-i(p_--p_+)})\psi_{k+1}$$
$$+(V_3 q^k + V_3^* q^{-k})\psi_k = E\psi_k, \tag{18}$$

where, by noting $\phi = P/Q$, we used a relation,

$$q = e^{i\pi\frac{P}{Q}} = e^{i\pi\phi}. \tag{19}$$

Inserting $z = q^k$ into the following functional equation and identifying $\psi_k = \psi(q^k)$,

$$(V_1 z q^{-1}e^{-i(p_-+p_+)} + V_2 z^{-1}e^{i(p_--p_+)})\psi(q^{-1}z) + (V_1 z^{-1}e^{i(p_-+p_+)} + V_2 zq e^{-i(p_--p_+)})\psi(qz)$$
$$+(V_3 z + V_3^* z^{-1})\psi(z) = E\psi(z), \tag{20}$$

we recover the original Schrödinger equation (18). In connection with our analysis below, it is crucial that a generic $\psi(z)$ is not always written in the form of an integral power in z of finite degree. In order to get such a special $\psi(z)$ of a finite polynomial in z, the parameters of the model need to be restricted.

Next we show that the Hamiltonian (8) with specific values of parameters can be described in terms of the generators of the extended $su_q(2)$. This is one of the cases in which the functional space consists of finite polynomials in z. In the diagonal gauge (13), we can write the Hamiltonian as a linear combination of generators of the extended $su_q(2)$.

We postulate the following form of Hamiltonian with undetermined parameters a and θ_α,

$$H = i(q - q^{-1})a(e^{-i\theta_\alpha} S_-(n_0) + e^{i\theta_\alpha} S_+(n_0)), \tag{21}$$

with positive $a > 0$ and real θ_α to maintain H hermitian in the standard $2j+1$ dimensional representation. Our H is a generalization of the Hamiltonian in Ref.[4] for $U_q(sl_2)$. Note that we always have $S_-(n_0) = S_+(n_0)^\dagger$ in the standard $2j+1$ dimensional representation if (7) is satisfied [15], whereas $S_- = S_+^\dagger$ is not ensured in general for $U_q(sl_2)$ with $n_0 = 0$.

On the functional space the Hamiltonian (21) acts as

$$H\Psi(z) = i(q - q^{-1})a(e^{-i\theta_\alpha} S_-(n_0) + e^{i\theta_\alpha} S_+(n_0))\Psi(z) = E\Psi(z). \tag{22}$$

The explicit form of this functional equation is written as (see eq.(4))

$$\begin{aligned} &ia(-e^{i\theta_\alpha} q^{1+n_0} z + e^{-i\theta_\alpha} q^{-n_0} z^{-1})\Psi(qz) + ia(e^{i\theta_\alpha} q^{-1-n_0} z - e^{-i\theta_\alpha} q^{n_0} z^{-1})\Psi(q^{-1}z) \\ &+ia(e^{-i\theta_\alpha} z^{-1} + e^{i\theta_\alpha} z)(q^{n_0} - q^{-n_0})\Psi(z) = E\Psi(z), \end{aligned} \tag{23}$$

where we assumed a very specific representation with $q^{2j+1} = 1$, which is satisfied by

$$2j + 1 = Q \quad and \quad P = even, \tag{24}$$

or

$$2j + 1 = 2Q, \tag{25}$$

for $q = e^{i\pi P/Q}$. Here we utilized the fact that $2j + 1$ and Q are generally independent integers. The choice $2j + 1 = Q$, which is also the choice in Ref. [4], suggests an integral j whereas $2j + 1 = 2Q$ suggests half an odd integer j. If we identify eq.(20) with eq.(23), we obtain

$$\begin{aligned} &V_1 e^{-i(p_- + p_+)} = iae^{i\theta_\alpha} q^{-n_0}, \\ &V_2 e^{i(p_- - p_+)} = -iae^{-i\theta_\alpha} q^{n_0}, \\ &V_3 = iae^{i\theta_\alpha}(q^{n_0} - q^{-n_0}), \\ &V_1 e^{i(p_- + p_+)} = iae^{-i\theta_\alpha} q^{-n_0}, \\ &V_2 e^{-i(p_- - p_+)} = -iae^{i\theta_\alpha} q^{+n_0}, \\ &V_3^* = iae^{-i\theta_\alpha}(q^{n_0} - q^{-n_0}). \end{aligned} \tag{26}$$

It can be confirmed that the conditions in (26) are satisfied by the following choice of parameters:

$$
\begin{aligned}
&V_1 = V_2 = a, \\
&p_+ = \pi m_1, \\
&p_- = -\theta_\alpha + \pi m_2, \\
&V_3 \equiv 2ae^{i\theta_3} = -2ae^{i\theta_\alpha}\sin(\pi n_0 P/Q),
\end{aligned} \tag{27}
$$

with suitable $m_1, m_2 \in Z$. In the last relation of these equations, we recalled $\sin \pi n_0 P/Q = \pm 1$ or $q^{n_0} = \pm i$ due to the definition in (7).

The extra potential terms in the Hamiltonian (8) are given by

$$
V_3 U + V_3^* U^\dagger = 4a \sum_{m,n} \cos\{\pi \frac{P}{Q}(m+n) + \theta_3\} c^\dagger_{m,n} c_{m,n}. \tag{28}
$$

This potential has a system size period $2Q$, and its periodicity is in the diagonal direction. The origin of the potential is shifted by the phase factor coming from V_3. (A way to realize this potential physically may be to super-impose an electric field, whose time variation is very slow.)

Now we construct Bethe ansatz equations. We know that the representation space of an extended $su_q(2)$ algebra is a polynomial of degree $2j$, namely, it can be expressed as

$$
\Psi(z) = \prod_{m=1}^{2j} (z - z_m). \tag{29}
$$

We come back to the original expression of our equation (23). By noting (29) we can rewrite this Eq.(23) as

$$
\begin{aligned}
&ia(-e^{i\theta_\alpha} q^{1+n_0} z + e^{-i\theta_\alpha} q^{-n_0} z^{-1}) \prod_{m=1}^{2j} \frac{qz - z_m}{z - z_m} \\
&+ia(e^{i\theta_\alpha} q^{-1-n_0} z - e^{-i\theta_\alpha} q^{n_0} z^{-1}) \prod_{m=1}^{2j} \frac{q^{-1}z - z_m}{z - z_m} \\
&+ia(e^{-i\theta_\alpha} z^{-1} + e^{i\theta_\alpha} z)(q^{n_0} - q^{-n_0}) = E.
\end{aligned} \tag{30}
$$

Let us assume that our representation is "irreducible" in the sense that all the $2j+1$ terms in polynomials of z appear in $\Psi(z)$ in (29), which amounts to all $z_m \neq 0$. The right-hand side of Eq.(30) is a constant and has no pole in z, then the left-hand side should also be

free of poles in z. By this pole free condition, we obtain Bethe ansatz equations

$$\frac{e^{2i\theta_\alpha} z_l^2 - q^{2n_0+1}}{q^{2n_0+1} e^{2i\theta_\alpha} z_l^2 + 1} = -\prod_{m=1}^{2j} \frac{q z_l - z_m}{z_l - q z_m}, \qquad l = 1, \cdots, 2j. \tag{31}$$

The energy eigenvalue is then given by

$$E = -iae^{i\theta_\alpha}(q^{n_0} - q^{-n_0} - q^{n_0-1} + q^{-n_0+1}) \sum_{m=1}^{2j} z_m, \tag{32}$$

by using the solutions of (31).

Finally, we discuss the possibility of writing the zero energy solution of Eq.(23) in terms of an Askey Wilson polynomial[20]. The Askey Wilson polynomial $P_m(w)$, which is a polynomial of degree m in $(w + w^{-1})$, is defined by

$$\begin{aligned} &A(w)P_m(q^2 w) + A(w^{-1})P_m(q^{-2}w) - (A(w) + A(w^{-1}))P_m(w) \\ &= (q^{-2m} - 1)(1 - abcdq^{2m-2})P_m(w), \end{aligned} \tag{33}$$

where

$$A(w) = \frac{(1 - aw)(1 - bw)(1 - cw)(1 - dw)}{(1 - w^2)(1 - q^2 w^2)}. \tag{34}$$

Let us set $a = -b = q$, $c = -d$ and replace q by $q^{1/2}$ in (33) ; c may depend on q and we replace $c(q) \to c(q^{1/2}) \equiv \bar{c}$. We then have

$$(1 - \bar{c}^2 w^2)P_m(qw) + (\bar{c}^2 - w^2)P_m(q^{-1}w) - (1 - w^2)(q^{-m} + \bar{c}^2 q^m)P_m(w) = 0. \tag{35}$$

We introduce the function $\Psi_m(w)$ which is a polynomial of degree $2m$ in w by

$$\Psi_m(w) = w^m P_m(w), \tag{36}$$

and replace w in (35) by $w = \xi z$. We then obtain

$$(1 - \bar{c}^2 \xi^2 z^2)\Psi_m(q\xi z) + (\bar{c}^2 - \xi^2 z^2)q^{2m}\Psi_m(q^{-1}\xi z) - (1 - \xi^2 z^2)(1 + \bar{c}^2 q^{2m})\Psi_m(\xi z) = 0. \tag{37}$$

If we equate (23) with $E = 0$ and (37) by identifying $\Psi(z) = \Psi_m(\xi z)$, we have the conditions

$$\begin{aligned} &\bar{c}^2 \xi^2 = e^{2i\theta_\alpha} q^{1+2n_0}, \\ &\bar{c}^2 q^{2m} = -q^{2n_0}, \\ &\xi^2 q^{2m} = -e^{2i\theta_\alpha} q^{-1}, \\ &1 + \bar{c}^2 q^{2m} = -(q^{2n_0} - 1), \\ &\xi^2(1 + \bar{c}^2 q^{2m}) = e^{2i\theta_\alpha}(q^{2n_0} - 1), \end{aligned} \tag{38}$$

which are solved by

$$\xi^2 = -e^{2i\theta_\alpha}, \quad \bar{c}^2 = -q^{1+2n_0}, \quad q^{2m} = q^{-1}, \tag{39}$$

for $q^{2n_0} \neq 1$ (Eq.(7) suggests $q^{2n_0} = -1$). The last relation in (39) gives $q^{2m+1} = 1$, which is consistent with our choice in (24) if $2m = 2j$. Namely we have $\Psi(z) = \Psi_j(\xi z)$, which is in fact a polynomial in z of degree $2j$. We have thus established that the zero energy solution of (23) for the choice of parameters in (24) is expressed by an Askey Wilson polynomial.

In passing we note that the limit $n_0 \to 0$ is usually singular, due to the condition (7), and therefore we cannot recover the results of Wiegmann and Zabrodin by a simple limit $n_0 \to 0$ of our final results. If one relaxes the condition (7) , on the other hand, the negative norm generally appears in the level of standard $2j+1$ dimensional representation. However, it appears that this does not necessarily imply the negative norm in the Bloch electron problem: Due to the non-unitary similarity transformation A, the state vectors in the Bloch electron problem correspond to $\langle b|A^{-1}$ and $A|k\rangle$, respectively, in terms of the bra $\langle b|$ and ket $|k\rangle$ vectors in the standard $2j+1$ dimensional representation, as was noted in connection with Eq.(5).

To summarize our consideration, the Hamiltonian of Bloch electrons in a uniform magnetic field with an additional periodic potential can be expressed as a linear combination of the generators of the extended q-deformed $su(2)$ algebra. The realization of this q-deformed $su(2)$ algebra on the functional space leads to functional Bethe ansatz equations. The solutions of the Bethe ansatz equations (31) are functions of finite degree for a generic value of n_0. It has also been shown that the zero energy solution is expressed by an Askey Wilson polynomial, but we have $e^{ip_+} = \pm 1$ instead of the mid-band condition $e^{ip_+} = i$ in [4].

One of the important aspects of the Bloch electron problem is multi-fractal behavior in the energy spectrum. In our Hamiltonian (21) with a generic n_0, the system could be at the off- critical point without the multi-fractal behavior[21] †. This property should be checked by a numerical diagonalization of the Schrödinger equation (18).

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